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Subclasses of typically real functions determined by some modular inequalities

Abstract. Let T be the family of all typically real functions, i.e. functions that are analytic in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and such that $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ for $z \in \Delta$. Moreover, let us denote: $\mathrm{T}^{(2)} := \{f \in \mathrm{T} : f(z) = -f(-z) \text{ for } z \in \Delta\}$ and $\mathrm{T}^{M,g} := \{f \in \mathrm{T} : f \prec Mg \text{ in } \Delta\}$, where $M > 1$, $g \in \mathrm{T} \cap \mathrm{S}$ and S consists of all analytic functions, normalized and univalent in Δ.

We investigate classes in which the subordination is replaced with the majorization and the function g is typically real but does not necessarily univalent, i.e. classes $\{f \in \mathrm{T} : f \ll Mg \text{ in } \Delta\}$, where $M > 1$, $g \in \mathrm{T}$, which we denote by $\mathrm{T}_{M,g}$. Furthermore, we broaden the class $\mathrm{T}_{M,g}$ for the case $M \in (0,1)$ in the following way: $\mathrm{T}_{M,g} = \{f \in \mathrm{T} : |f(z)| \geq M|g(z)| \text{ for } z \in \Delta\}$, $g \in \mathrm{T}$.

1. Introduction. Let T be the family of all typically real functions, i.e. functions that are analytic in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0)-1 = 0$ and such that $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ for $z \in \Delta$. Let S denote the class of all analytic functions, normalized as above and univalent in Δ, and SR – the subclass of S consisting of functions with real coefficients. Moreover, let us denote: $\mathrm{T}^{(2)} := \{f \in \mathrm{T} : f(z) = -f(-z) \text{ for } z \in \Delta\}$ and $\mathrm{T}^{M,g} := \{f \in \mathrm{T} : f \prec Mg \text{ in } \Delta\}$, where $M > 1$, $g \in \mathrm{T} \cap \mathrm{S}$. The symbol $h \prec H$ denotes the subordination in Δ, i.e. $h(0) = H(0)$ and $h(\Delta) \subset H(\Delta)$, where H is univalent. Let us notice that for $g_1(z) = z$

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and $g_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ we have $\mathrm{T}^{M,g_1} = \{f \in \mathrm{T} : |f| < M \text{ in } \Delta\}$ and $\mathrm{T}^{M,g_2} = \{f \in \mathrm{T} : |\operatorname{Im} f| < M\pi/4 \text{ in } \Delta\}$, $M > 1$. These classes are briefly denoted by T_M and $\mathrm{T}(M)$, respectively.

The subordination in the classes T, S and SR has been investigated by several authors (for example [2], [3], [4]). The relation $\mathrm{T}^{M,g} = \{Mg\,(h/M) : h \in \mathrm{T}_M\}$ for $g \in \mathrm{T} \cap \mathrm{S}$ (see [3]) provides the following formula connecting different classes of type $\mathrm{T}^{M,g}$: $\mathrm{T}^{M,f} = \left\{Mf\left(g^{-1}\left(h/M\right)\right) : h \in \mathrm{T}^{M,g}\right\}$, $f, g \in \mathrm{T} \cap \mathrm{S}$. For this reason, instead of researching a class $\mathrm{T}^{M,f}$ one can consider a class $\mathrm{T}^{M,g}$, for instance T_M or $\mathrm{T}(M)$. We apply this idea to obtain results in various classes $\mathrm{T}^{M,g}$ from corresponding results in the class $\mathrm{T}(M)$. Investigating $\mathrm{T}(M)$ is possible because the integral formula for this class, the set of extremal points and the set of supporting points are known (see [4]).

Moreover, it is easy to prove that the class $\mathrm{T}^{M,g} \cap \mathrm{T}^{(2)} = \{Mg\,(h/M) : h \in \mathrm{T}_M\}$ for $g \in \mathrm{T}^{(2)} \cap \mathrm{S}$.

In the paper we investigate classes similar to $\mathrm{T}^{M,g}$, in which the subordination is replaced with the majorization (the modular subordination) and the function g is typically real but does not necessarily univalent, i.e. classes $\mathrm{T}_{M,g} := \{f \in \mathrm{T} : f \ll Mg \text{ in } \Delta\}$, where $M > 1$, $g \in \mathrm{T}$. The symbol $h \ll H$ denotes the majorization in Δ, i.e. $|h(z)| \leq |H(z)|$ for all $z \in \Delta$.

Furthermore, we broaden the class $\mathrm{T}_{M,g}$ for the case when $M \in (0,1)$ in the following way: $\mathrm{T}_{M,g} = \{f \in \mathrm{T} : |f(z)| \geq M|g(z)| \text{ for } z \in \Delta\}$, $g \in \mathrm{T}$.

Moreover, we study the subclass of the class $\mathrm{T}_{M,g}$, consisting of all odd functions, which we denote by $\mathrm{T}^{(2)}_{M,g}$.

The class $\mathrm{T}_{M,g}$ is not empty, because for example the function g belongs to this class. Analogously, the class $\mathrm{T}^{(2)}_{M,g}$ for $g \in \mathrm{T}^{(2)}$ is not empty. If $M = 1$, then the class consists of only one function g. So we investigate the class $\mathrm{T}_{M,g}$ for $M \in (0,1) \cup (1,\infty)$. For $g = id$ and $M \geq 1$, we have $\mathrm{T}^{M,id} = \mathrm{T}_{M,id}$.

In the class $\mathrm{T}^{M,g}$ one can formulate theorems which are true for each function $g \in \mathrm{T} \cap \mathrm{S}$. However, in the class $\mathrm{T}_{M,g}$ it is impossible. Indeed, theorems in the class $\mathrm{T}_{M,g}$ in a fundamental way depends on the choice of the function g. It means that a theorem which is true in the class T_{M,g_1} generally is not true in the class T_{M,g_2}, for $g_1 \neq g_2$. In each case, we connect the researching class with the class T_M or $\mathrm{T}^{(2)}_M$.

2. Some properties of the classes T and $\mathbf{T}^{(2)}$. During our investigation of the class $\mathrm{T}_{M,g}$, we use the following relations of classes T and $\mathrm{T}^{(2)}$, which we give as lemmas. In each lemma we shall prove only one implication. The other can be proved analogously. For simplicity, instead of h or $z \mapsto h(z)$ we will use $h(z)$.

Lemma 1. $f \in \mathrm{T} \iff \frac{1+z^2}{z} f(z^2) \in \mathrm{T}^{(2)}$.

Proof. Let $f \in \mathrm{T}$. For $f \in \mathrm{T}$ we have the Robertson formula $f(z) = \int_{-1}^{1} \frac{z}{1-2zt+z^2} d\mu(t)$, where μ is a probability measure on $[-1,1]$ (see [1], [2]). Then

$$\frac{(1+z^2)f(z^2)}{z} = \int_{-1}^{1} \frac{z(1+z^2)}{1-2z^2t+z^4} d\mu(t) = \int_{-1}^{1} \frac{z(1+z^2)}{(1+z^2)^2-2(1+t)z^2} d\mu(t)$$
$$= \int_{0}^{1} \frac{z(1+z^2)}{(1+z^2)^2-4\tau z^2} d\nu(\tau)$$

with $\nu(A) \equiv \mu(2A-1)$ (where A is a Borel set contained in $[0,1]$). Clearly, $\int_0^1 \frac{z(1+z^2)}{(1+z^2)^2-4\tau z^2} d\nu(\tau) \in \mathrm{T}^{(2)}$ (the representation formula for functions from the class $\mathrm{T}^{(2)}$, see [5]). Therefore, $\frac{(1+z^2)f(z^2)}{z} \in \mathrm{T}^{(2)}$. $\square$

Lemma 2. $f \in \mathrm{T}^{(2)} \iff \frac{1+z^2}{1-z^2}\frac{f(iz)}{i} \in \mathrm{T}^{(2)}$.

Proof. Suppose that $f \in \mathrm{T}^{(2)}$. From Lemma 1, the function h given by $h(z^2) = \frac{z}{1+z^2} f(z)$ is in T. The definition of h is correct since $h\left((-z)^2\right) = \frac{-z}{1+(-z)^2} f(-z) = \frac{zf(z)}{1+z^2} = h(z^2)$. Then $f(iz) = \frac{1-z^2}{iz} h(-z^2)$. Hence, $\frac{1+z^2}{1-z^2}\frac{f(iz)}{i} = -\frac{1+z^2}{z} h(-z^2)$. Because of Lemma 1 and the fact that $h \in \mathrm{T} \Leftrightarrow -h(-z) \in \mathrm{T}$, we receive $-\frac{1+z^2}{z}h(-z^2) \in \mathrm{T}^{(2)}$. This means that $\frac{1+z^2}{1-z^2}\frac{f(iz)}{i} \in \mathrm{T}^{(2)}$, so we have the desired result. $\square$

Lemma 3. $f \in \mathrm{T} \iff \frac{z^2}{(1-z^2)^2}\frac{1}{f(z)} \in \mathrm{T}$.

Proof. Let $f \in \mathrm{T}$. Then $f(z) = \frac{z}{1-z^2} p(z)$ for $p \in \mathrm{PR}$ (the Rogosinski representation, [2], [6]), where PR consists of all analytic functions p such that $p(0) = 1$, $\mathrm{Re}\, p(z) > 0$ for $z \in \Delta$ and having real coefficients. Clearly, $\frac{1}{p} \in \mathrm{PR}$, so $\frac{z}{1-z^2}\frac{1}{p(z)} \in \mathrm{T}$, i.e. $\frac{z^2}{(1-z^2)^2}\frac{1}{f(z)} \in \mathrm{T}$. From this and the equality $\left\{\frac{1}{p} : p \in \mathrm{PR}\right\} = \mathrm{PR}$, we get $f \in \mathrm{T} \Leftrightarrow \frac{z^2}{(1-z^2)^2}\frac{1}{f(z)} \in \mathrm{T}$. $\square$

Taking $f \in \mathrm{T}^{(2)}$ in Lemma 3, we obtain the following relation:

Lemma 4. $f \in \mathrm{T}^{(2)} \iff \frac{z^2}{(1-z^2)^2}\frac{1}{f(z)} \in \mathrm{T}^{(2)}$.

Lemma 5. $f \in \mathrm{T} \iff \frac{z^3}{(1-z^4)(1-z^2)}\frac{1}{f(z^2)} \in \mathrm{T}^{(2)}$.

Proof. Let $f \in \mathrm{T}$. On the basis of Lemma 1, the function g given by $g(z) = \frac{1+z^2}{z} f(z^2)$ belongs to $\mathrm{T}^{(2)}$. Hence, we have $\frac{z^2}{(1-z^2)^2}\frac{1}{g(z)} = \frac{z^3}{(1-z^4)(1-z^2)}\frac{1}{f(z^2)}$. From Lemma 4, we know that $\frac{z^2}{(1-z^2)^2}\frac{1}{g(z)} \in \mathrm{T}^{(2)}$ which is equivalent to $\frac{z^3}{(1-z^4)(1-z^2)}\frac{1}{f(z^2)} \in \mathrm{T}^{(2)}$. $\square$

Lemma 6. $f \in \mathrm{T}^{(2)} \iff \frac{z^2}{1-z^4}\frac{i}{f(iz)} \in \mathrm{T}^{(2)}$.

Proof. Suppose that $f \in \mathrm{T}^{(2)}$. Let $g(z) = \frac{1+z^2}{1-z^2}\frac{f(iz)}{i}$. By Lemma 2, $g \in \mathrm{T}^{(2)}$. Since $\frac{z^2}{(1-z^2)^2}\frac{1}{g(z)} = \frac{z^2}{1-z^4}\frac{i}{f(iz)}$, from Lemma 4 we get $\frac{z^2}{(1-z^2)^2}\frac{1}{g(z)} \in \mathrm{T}^{(2)}$ i.e. $\frac{z^2}{1-z^4}\frac{i}{f(iz)} \in \mathrm{T}^{(2)}$. $\square$

3. The majorization in the class of typically real functions T. At the beginning we study the case when $M > 1$, i.e. the class

$$\mathrm{T}_{M,g} = \{f \in \mathrm{T} : |f(z)| \leq M|g(z)| \text{ for } z \in \Delta\}, \quad g \in \mathrm{T}.$$

At first, let $g(z) = \frac{z}{1+z}$. Clearly, $g \in \mathrm{T} \cap \mathrm{S}$.

Theorem 1. *If $f \in \mathrm{T}$ and $|f(z)| \leq M\left|\frac{z}{1+z}\right|$ for all $z \in \Delta$, $M > 1$ (i.e. $f \in \mathrm{T}_{M,g}$ where $g(z) = \frac{z}{1+z}$), then $f(z^2) \equiv \frac{z}{1+z^2}h(z)$ for some $h \in \mathrm{T}^{(2)}_M$.*

Proof. Let $f \in \mathrm{T}$ and $|f(z)| \leq M\left|\frac{z}{1+z}\right|$. Hence, $|f(z^2)| \leq M\left|\frac{z^2}{1+z^2}\right|$. Let $h(z) \equiv \frac{1+z^2}{z}f(z^2)$. By Lemma 1, $h \in \mathrm{T}^{(2)}$. Therefore, $f(z^2) \equiv \frac{z}{1+z^2}h(z)$. From the above equality, we get $\left|\frac{z}{1+z^2}\right||h(z)| \leq M\left|\frac{z^2}{1+z^2}\right|$. This implies that $|h(z)| \leq M|z| < M$, that is $h \in \mathrm{T}^{(2)}_M$. $\square$

Now, let us consider the function $g(z) = z + z^3$. We have $g(z) = \frac{z}{1-z^2}(1-z^4)$. Since $\mathrm{Re}(1-z^4) > 0$ for $z \in \Delta$, from the Rogosinski formula (see [2], [6]), we get $g \in \mathrm{T}$. Moreover, $g \in \mathrm{T}^{(2)}$ and $g \notin \mathrm{S}$, because $g'\left(i/\sqrt{3}\right) = 0$.

Theorem 2. *If $f \in \mathrm{T}^{(2)}$ and $|f(z)| \leq M|z+z^3|$ for all $z \in \Delta$, $M > 1$ (i.e. $f \in \mathrm{T}^{(2)}_{M,g}$, where $g(z) = z + z^3$), then $f(z) \equiv \frac{1+z^2}{z}h(z^2)$ for some $h \in \mathrm{T}_M$.*

Proof. Suppose that $f \in \mathrm{T}^{(2)}$ and $|f(z)| \leq M|z+z^3|$. By Lemma 1, the function h given by $h(z^2) \equiv \frac{z}{1+z^2}f(z)$ is in T. Therefore, $f(z) \equiv \frac{1+z^2}{z}h(z^2)$. From the second assumption, we have $\left|\frac{1+z^2}{z}\right||h(z^2)| \leq M|z+z^3|$. Then $|h(z^2)| \leq M|z^2| < M$, i.e. $h \in \mathrm{T}_M$. $\square$

Let us study the next function $g(z) = \frac{z+z^3}{1-z^2}$. We have $g(z) = \frac{z}{1-z^2}(1+z^2)$. Since $\mathrm{Re}(1+z^2) > 0$ for $z \in \Delta$, from the Rogosinski formula, $g \in \mathrm{T}$. Furthermore, $g \in \mathrm{T}^{(2)}$ and $g \notin \mathrm{S}$, because $g'\left(\sqrt{\sqrt{5}-2}\; i\right) = 0$.

Theorem 3. *If $f \in \mathrm{T}^{(2)}$ and $|f(z)| \leq M\left|\frac{z+z^3}{1-z^2}\right|$ for all $z \in \Delta$, $M > 1$ (i.e. $f \in \mathrm{T}^{(2)}_{M,g}$ where $g(z) = \frac{z+z^3}{1-z^2}$), then $f(z) \equiv \frac{1+z^2}{1-z^2}\frac{h(iz)}{i}$ for some $h \in \mathrm{T}^{(2)}_M$.*

Proof. Assume that $f \in \mathrm{T}^{(2)}$ and $|f(z)| \leq M\left|\frac{z+z^3}{1-z^2}\right|$. Let $h(iz) \equiv \frac{1-z^2}{1+z^2}if(z)$. By Lemma 2, $h \in \mathrm{T}^{(2)}$. Hence, $f(z) \equiv \frac{1+z^2}{1-z^2}\frac{h(iz)}{i}$. From the above equality, we get $\left|\frac{1+z^2}{1-z^2}\right||h(iz)| \le M\left|\frac{z+z^3}{1-z^2}\right|$. Therefore, $|h(iz)| \le M|z| < M$, that is $h \in \mathrm{T}^{(2)}_M$. □

In the further investigation we consider the case when $M \in (0,1)$, i.e. the class

$$\mathrm{T}_{M,g} = \{f \in \mathrm{T} : |f(z)| \ge M|g(z)| \text{ for } z \in \Delta\}, \quad g \in \mathrm{T}.$$

Suppose that $g(z) = \frac{z}{(1-z^2)^2}$. Since $g(z) = \frac{z}{1-z^2}\frac{1}{1-z^2}$ and $\mathrm{Re}\left(\frac{1}{1-z^2}\right) > 0$ for $z \in \Delta$, hence $g \in \mathrm{T}$. We have also $g'(i/\sqrt{3}) = 0$, and it follows that $g \notin \mathrm{S}$.

Theorem 4. *If $f \in \mathrm{T}$ and $|f(z)| \ge M\left|\frac{z}{(1-z^2)^2}\right|$ for all $z \in \Delta$, $M \in (0,1)$ (i.e. $f \in \mathrm{T}_{M,g}$ where $g(z) = \frac{z}{(1-z^2)^2}$), then $f(z) \equiv \frac{z^2}{(1-z^2)^2}\frac{1}{h(z)}$ for some $h \in \mathrm{T}_{1/M}$.*

Proof. Let $f \in \mathrm{T}$ and $|f(z)| \ge M\left|\frac{z}{(1-z^2)^2}\right|$. By Lemma 3, the function h given by $h(z) \equiv \frac{z^2}{(1-z^2)^2}\frac{1}{f(z)}$ belongs to T. So $f(z) \equiv \frac{z^2}{(1-z^2)^2}\frac{1}{h(z)}$. From the second assumption, we have $\left|\frac{z^2}{(1-z^2)^2}\right|\frac{1}{|h(z)|} \ge M\left|\frac{z}{(1-z^2)^2}\right|$ i.e. $|h(z)| \le |z|/M < 1/M$. Hence, $h \in \mathrm{T}_{1/M}$ and the proof is complete. □

Analogously, using Lemma 4, we prove the following theorem:

Theorem 5. *If $f \in \mathrm{T}^{(2)}$ and $|f(z)| \ge M\left|\frac{z}{(1-z^2)^2}\right|$ for all $z \in \Delta$, $M \in (0,1)$ (i.e. $f \in \mathrm{T}^{(2)}_{M,g}$ where $g(z) = \frac{z}{(1-z^2)^2}$), then $f(z) \equiv \frac{z^2}{(1-z^2)^2}\frac{1}{h(z)}$ for some $h \in \mathrm{T}^{(2)}_{1/M}$.*

Now, let us consider the function $g(z) = \frac{z}{(1-z^2)(1-z)}$. Clearly, $g(z) = \frac{z}{1-z^2}\frac{1}{1-z}$ and $\mathrm{Re}\left(\frac{1}{1-z}\right) > 0$ for $z \in \Delta$, so $g \in \mathrm{T}$. We have also

$$g'\left((i\sqrt{7}-1)/4\right) = 0,$$

which means that $g \notin \mathrm{S}$.

Theorem 6. *If $f \in \mathrm{T}$ and $|f(z)| \ge M\left|\frac{z}{(1-z^2)(1-z)}\right|$ for all $z \in \Delta$, $M \in (0,1)$ (i.e. $f \in \mathrm{T}_{M,g}$ where $g(z) = \frac{z}{(1-z^2)(1-z)}$), then $f(z^2) \equiv \frac{z^3}{(1-z^4)(1-z^2)}\frac{1}{h(z)}$ for some $h \in \mathrm{T}^{(2)}_{1/M}$.*

Proof. Suppose that $f \in \mathrm{T}$ and $|f(z)| \ge M\left|\frac{z}{(1-z^2)(1-z)}\right|$. By Lemma 5, the function $h(z) \equiv \frac{z^3}{(1-z^4)(1-z^2)}\frac{1}{f(z^2)}$ is in $\mathrm{T}^{(2)}$. Hence, $f(z^2) \equiv \frac{z^3}{(1-z^4)(1-z^2)}\frac{1}{h(z)}$. From the second assumption, we get $\left|\frac{z^3}{(1-z^4)(1-z^2)}\right|\frac{1}{|h(z)|} \ge M\left|\frac{z^2}{(1-z^4)(1-z^2)}\right|$, so $|h(z)| \leq |z|/M < 1/M$. This means that $h \in \mathrm{T}^{(2)}_{1/M}$, so we have the desired result. $\square$

Now let us study the function $g(z) = \frac{z}{1-z^4}$. Because $g(z) = \frac{z}{1-z^2}\frac{1}{1+z^2}$ and $\mathrm{Re}\left(\frac{1}{1+z^2}\right) > 0$ for $z \in \Delta$, so $g \in \mathrm{T}$. Moreover, $g \in \mathrm{T}^{(2)}$ and $g \notin \mathrm{S}$, because $g'\left((i+1)/\sqrt[4]{12}\right) = 0$.

Theorem 7. *If $f \in \mathrm{T}^{(2)}$ and $|f(z)| \geq M\left|\frac{z}{1-z^4}\right|$ for all $z \in \Delta$, $M \in (0,1)$ (i.e. $f \in \mathrm{T}^{(2)}_{M,g}$ where $g(z) = \frac{z}{1-z^4}$), then $f(iz) \equiv \frac{z^2}{1-z^4}\frac{i}{h(z)}$ for some $h \in \mathrm{T}^{(2)}_{1/M}$.*

Proof. Let $f \in \mathrm{T}^{(2)}$ and $|f(z)| \geq M\left|\frac{z}{1-z^4}\right|$. By Lemma 6, the function $h(z) \equiv \frac{z^2}{1-z^4}\frac{i}{f(iz)}$ belongs to $\mathrm{T}^{(2)}$. So $f(iz) \equiv \frac{z^2}{1-z^4}\frac{i}{h(z)}$. From the second assumption, we have $\left|\frac{z^2}{1-z^4}\right|\frac{1}{|h(z)|} \geq M\left|\frac{iz}{1-z^4}\right|$ i.e. $|h(z)| \leq |z|/M < 1/M$. Therefore, $h \in \mathrm{T}^{(2)}_{1/M}$ and the proof is complete. $\square$

The converses to Theorems 1–7 are also true.

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